Barrier Put-Call Transformations *

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February 16, 1999

*Iniciated March 1997. Presented at the Danske Bank Symposium on Securities with Embedded Options 1998.

†I would like to thank Jørgen Haug for helpful comments on this paper.

Abstract

In this article we show a simple but important relationship between the put-call transformation and the put-call symmetry as well as extend the relationship to also hold for single and double barrier options. These new barrier transformations give new insight in barrier option valuation. Using the transformation it is possible to value a barrier put option from a barrier call option formula and vice versa. Our results also extend the possibilities for static hedging and closed form valuation for many new exotic options. The new relationships also make us able to value a double barrier option in a simple and intuitive way, only using a few single barrier options.

1 Plain vanilla put-call transformation

Assume the underlying asset follows a geometric Brownian motion $dS = \mu S dt + \sigma S dz$, where as usual μ is the expected instantaneous rate of return on the underlying asset, σ is the instantaneous standard deviation of the rate of return, and dz is a standard Wiener process. Given this the American plain vanilla put-call transformation, first published by Bjerksund and Stensland (1993), states that

$$C(S, X, T, r, b, \sigma) = P(X, S, T, r - b, -b, \sigma), \tag{1}$$

Where S is the asset price, X the strike price, T time to maturity, r the risk free rate, and b the cost of carry. In other words the value of an American call option is similar to the value of an American put option with asset price equal to the strike price, strike equal to the asset price, risk-free rate equal to $r-b$ and cost of carry equal to $-b$. This relationship naturally also holds for European options. The put-call transformation gives insight in the connection between put and call options and makes us able to price a put option from the formula of a call option and vice versa.

However, the usefulness of this transformation for static hedging and valuation of a large class of exotic options has first recently come to attention in the form of the so-called put-call symmetry. We can easily rewrite the payoff function from a call (or similarly for a put) option, $\max(S - X, 0)$, into $\frac{X}{S}\max\left(\frac{S^2}{X} - S, 0\right)$. Then by combining this with the put-call transformation we simply get the put-call symmetry

$$C(S, X, T, r, b, \sigma) = \frac{X}{S} P\left(S, \frac{S^2}{X}, T, r - b, -b, \sigma\right), \tag{2}$$

Thus the more recently published European and American put-call symmetry (see Carr and Bowie (1994), Carr, Ellis, and Gaupta (1998), Carr and Chesney (1998), Haug (1997)) is actually no more than a simple rewriting of the put-call transformation. However, for practitioners/financial engineers this small

rewriting of the put-call transformation is actually very useful because it is first in the symmetry form the put-call transformation really becomes useful for static hedging and valuation of many exotic options on the basis of plain vanilla options[1]. The main reason for this is that it is not possible to buy for instance a put option with asset price X when the asset price is S (assuming $X <> S$). To buy $\frac{X}{S}$ number of put options with strike $\frac{S^2}{X}$ and asset price S is on the other hand a real possibility in the options market[2].

2 Barrier Put-Call Transformation

Barrier options have become extremely popular and certainly constitutes one of the most popular class of exotic options. Closed form solutions for valuation of single barrier options have been published by Merton (1973), Reiner and Rubinstein (1991), and Rich (1994), and for double barrier options by Ikeda and Kuintomo (1992), and Geman and Yor (1996). Further, the relationship between in and out options are well known as the in-out barrier parity. A long out option is equal to a long plain vanilla option plus short an in option.

in this paper we go one step further and state the put-call transformation for European and American single and double barrier options. Given that the plain vanilla put-call transformation holds the intuition behind the put-call barrier transformation is quite intuitive. The only difference between a plain vanilla put-call transformation and a put-call barrier transformation is the probability of barrier hits. Given the same volatility and drift towards the barrier the probability of barrier hits only depend on the distance between the asset price and the barrier. In the put call transformation the drifts are different on the call and the put, b versus $-b$. However, given that the asset price of the call is above (below) the barrier and the asset price of the put is below (above) the barrier this will naturally ensure the same drift towards the barrier. In the case of a put-call transformation between a down-call with asset price S and an up-put with asset price X it must be that

$$\ln\left(\frac{S}{H_c}\right) = \ln\left(\frac{H_p}{X}\right) \tag{3}$$

where the call barrier $H_c < S$ and the put barrier $H_p > X$. In the case of a put-call transformation between an up-call and a down-put the barriers and strike must satisfy;

$$\ln\left(\frac{H_c}{S}\right) = \ln\left(\frac{X}{H_p}\right) \tag{4}$$

where $H_c > S$ and $H_p < X$. In both cases we can rewrite the put barrier as $\frac{SX}{H_c}$. For standard barrier options the put-call transformation and symmetry between in-options must from this be given by

[1] It is worth mentioning that when used for static hedging of most path dependent options the published put-call transformation only holds when cost-of-carry is zero (i.e. options on futures and forwards). This is because one doesn't know exactly when the barrier will be hit.

[2] At least in most OTC option markets.

$$\begin{aligned} C_{di}(S,X,H,r,b) &= P_{ui}\left(X,S,\frac{SX}{H},r-b,-b\right) \\ &= \frac{X}{S}P_{ui}\left(S,\frac{S^2}{X},\frac{S^2}{H},r-b,-b\right) \end{aligned} \tag{5}$$

$$\begin{aligned} C_{ui}(S,X,H,r,b) &= P_{di}\left(X,S,\frac{SX}{H},r-b,-b\right) \\ &= \frac{X}{S}P_{di}\left(S,\frac{S^2}{X},\frac{S^2}{H},r-b,-b\right) \end{aligned} \tag{6}$$

where C_{di} stands for a down-and-in call, and C_{ui} stands for a up-and-in call (similarly for puts). The put-call transformation between out barrier options is given by:

$$\begin{aligned} C_{do}(S,X,H,r,b) &= P_{uo}\left(X,S,\frac{SX}{H},r-b,-b\right) \\ &= \frac{X}{S}P_{uo}\left(S,\frac{S^2}{X},\frac{S^2}{H},r-b,-b\right) \end{aligned} \tag{7}$$

$$\begin{aligned} C_{uo}(S,X,H,r,b) &= P_{do}\left(X,S,\frac{SX}{H},r-b,-b\right) \\ &= \frac{X}{S}P_{do}\left(S,\frac{S^2}{X},\frac{S^2}{H},r-b,-b\right) \end{aligned} \tag{8}$$

and for double barrier options we have:

$$\begin{aligned} C_{o}(S,X,L,U,r,b) &= P_{o}\left(X,S,\frac{SX}{U},\frac{SX}{L},r-b,-b\right) \\ &= \frac{X}{S}P_{o}\left(S,\frac{S^2}{X},\frac{S^2}{U},\frac{S^2}{L},r-b,-b\right) \end{aligned} \tag{9}$$

$$\begin{aligned} C_{i}(S,X,L,U,r,b) &= P_{i}\left(X,S,\frac{SX}{U},\frac{SX}{L},r-b,-b\right) \\ &= \frac{X}{S}P_{i}\left(S,\frac{S^2}{X},\frac{S^2}{U},\frac{S^2}{L},r-b,-b\right) \end{aligned} \tag{10}$$

where L is the lower barrier and U is the upper barrier level. These transformations also hold for partial-time single and double barrier options described by Heynen and Kat (1994) and Hui (1997).

These new transformation/symmetry relationships give new insight and should be useful when calculating barrier option values. If one has a formula for a barrier call, the relationship will give the value for the barrier put and vice versa. The relationship also gives new opportunities for static hedging and valuation of 2. generation exotic options. An example of this would be a first-down-then-up-and-in call. In a first-down-then-up-and-in call $C_{dui}(S, X, L, U)$ the option holder gets a standard up-and-in call with barrier $U(U > S)$ and strike X if the asset first hit a lower barrier $L(L < S < U)$. Using the up-and-in call down-and-in put barrier symmetry described above we can simply construct a static hedge and thereby a valuation formula for this new type of barrier option;

$$C_{dui}(S, X, L, U) = \frac{X}{L} P_{di}\left(S, \frac{L^2}{X}, \frac{L^2}{U}\right) \tag{11}$$

In other words to hedge a first-down-then-up-and-in barrier call option all we need to do is buying $\frac{X}{L}$ number of standard down-and-in puts with strike $\frac{L^2}{X}$ and barrier $\frac{L^2}{U}$. If the asset price never touche L both the first-down-then-up-and-in call and the standard down-and-in put will expire worthless. On the other hand, if the asset price hits the lower barrier L the value of the $\frac{X}{L}$ down-and-in puts will be exactly equal to the value of the up-and-in call. So in that case all we need to do is sell the down-and-in put and simultaneously buy the up-and-in call. As we can see we have created a perfect static hedge for this new barrier option using only standard barrier options and the barrier transformation principle.

In a similar fashion one can easily construct static hedges and valuation formulas for a large class of new barrier options.

3 Simple, intuitive and accurate valuation of double barrier options

Ikeda and Kuintomo (1992), and Geman and Yor (1996) have developed closed form formulas for double barrier options using quite complex mathematics.

An alternative is to value double barrier options using the single barrier put-call transformations in combination with some simple intuition. The idea can best be illustrated by first trying to construct a static hedge for a double barrier option by using only single barrier options. Let's assume we want to try to statically hedge a double barrier knock-in-call option with lower barrier L, upper barrier U, and strike price X.

A natural first step could be to buy an up-and-in-call and a down-and-in-call. As long as the asset not touch any of the two barriers, or only touch one of the barriers this hedge works fine. The problem with this strategy is naturally that if the asset touch both barriers we end up with two call options instead of one. In other words we are over hedged. This makes our static hedge unnecessary expensive.

To avoid this we could simply add a short position in a first-up-then-down-and-in-call and a first-down-then-up-and-in-call. In this case when the asset for the

first time hits a barrier we get a call. Then if the asset should hit the other barrier we get two new options at the same time; a long call plus a short call which cancel each other out. We are only left with the call we got at the first barrier hit as we should.

The problem with this strategy occurs if the asset first hits the upper barrier, then hit the lower barrier, and then hits the upper barrier. Or alternatively, if the asset first hits the lower barrier, then hits the upper barrier, and then hits the lower barrier. In any of these cases we will end up with zero call options. To avoid this we could simply add a long position in a first-down-then-up-then-down-and-in call plus a first-up-then-down-then-up-and-in call.

Now we will first get into trouble if the asset first hits the upper barrier, then the lower barrier, then the upper barrier and then the lower barrier. Or alternatively if the asset first hits the lower barrier, then hits the upper barrier, then hits the lower barrier, and then hits the upper barrier. To avoid this we could simply add a short position in a first-up-then-down-then-up-then-down-and-in and a first-down-then-up-then-down-then-up-and-in call.

Continuing this way one will soon find that a double barrier option is nothing more than an infinite series of the new type of barrier options introduced in section 2. Since these new types of barrier options (e.g. a first-down-then-up-then-down...and-in) can be constructed simply by using the single barrier put-call transformation, a double barrier option is nothing more than an infinite series of single barrier options;

$$\text{Double barrier option} = \sum_{i=1}^{\infty} \text{Single barrier options} \tag{12}$$

To value an infinite series is naturally umpractical if not impossible. However the probability of the asset first touching the upper barrier then touching the lower barrier, then touching the upper barrier, then touching the lower barrier.....then touching the upper barrier is in most cases fast getting extremely small. Numerical investigation shows that our method converges very fast in most cases only using the first 3 or 4 correction terms. Using four terms, the value of a double barrier in-call option can be approximated by

$$\begin{aligned} C_i(S,X,L,U) \approx\ & C_{ui}(S,X,U) + C_{di}(S,X,L) \\ & -\frac{X}{U}P_{ui}\left(S,\frac{U^2}{X},\frac{U^2}{L}\right) - \frac{X}{L}P_{di}\left(S,\frac{L^2}{X},\frac{L^2}{U}\right) \\ & +\frac{U}{L}C_{di}\left(S,\frac{L^2X}{U^2},\frac{L^3}{U^2}\right) + \frac{L}{U}C_{ui}\left(S,\frac{U^2X}{L^2},\frac{U^3}{L^2}\right) \\ & -\frac{LX}{U^2}P_{ui}\left(S,\frac{U^4}{L^2X},\frac{U^4}{L^3}\right) - \frac{UX}{L^2}P_{di}\left(S,\frac{L^4}{U^2X},\frac{L^4}{U^3}\right) \\ & +\frac{U^2}{L^2}C_{di}\left(S,\frac{L^4X}{U^4},\frac{L^5}{U^4}\right) + \frac{L^2}{U^2}C_{ui}\left(S,\frac{U^4X}{L^4},\frac{U^5}{L^4}\right) \end{aligned} \tag{13}$$

and similarly the value of a double barrier in-put can be approximated by

$$
\begin{aligned}
P_i(S,X,L,U) \quad \approx \quad & P_{ui}(S,X,U) + P_{di}(S,X,L) \\
& -\frac{X}{U}C_{ui}\left(S,\frac{U^2}{X},\frac{U^2}{L}\right) - \frac{X}{L}C_{di}\left(S,\frac{L^2}{X},\frac{L^2}{U}\right) \\
& +\frac{U}{L}P_{di}\left(S,\frac{L^2X}{U^2},\frac{L^3}{U^2}\right) + \frac{L}{U}P_{ui}\left(S,\frac{U^2X}{L^2},\frac{U^3}{L^2}\right) \\
& -\frac{LX}{U^2}C_{ui}\left(S,\frac{U^4}{L^2X},\frac{U^4}{L^3}\right) - \frac{UX}{L^2}C_{di}\left(S,\frac{L^4}{U^2X},\frac{L^4}{U^3}\right) \\
& +\frac{U^2}{L^2}P_{di}\left(S,\frac{L^4X}{U^4},\frac{L^5}{U^4}\right) + \frac{L^2}{U^2}P_{ui}\left(S,\frac{U^4X}{L^4},\frac{U^5}{L^4}\right)
\end{aligned} \tag{14}
$$

The value of double barrier out-options can easily be found using the out-in barrier parity.

Table 1 shows the difference in value between the Ikeda and Kuintomo (1992)[3] model and our approximation.

Table 1: The Ikeda and Kunitomo model minus our intuitive double barrier model (using 4 terms in our model and 20 leading terms in the Ikeda and Kuntimo formula), ($S = 100$, $X = 100$, $r = 0.10$, $b = 0$)

		$T = 0.25$			$T = 1$		
L	U	$\sigma = 10\%$	$\sigma = 20\%$	$\sigma = 30\%$	$\sigma = 10\%$	$\sigma = 20\%$	$\sigma = 30\%$
50	150	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
60	140	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
70	130	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
80	120	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
95	105	0.000000	0.000000	-0.001244	0.000000	-0.032843	-0.518185

From all the zeros in the table we can see that the two models must give almost identical values for most parameters. First when the volatility is quite high in combination with long time to maturity, and the difference between the lower and upper barrier are quite small our model gets into trouble. The reason for this is simply because in that case the probability for the asset price to hit the lower and upper barriers many times in succession increases. Using only 4 terms our model will get into trouble if the asset goes back and forth between the lower and upper barriers more than 4 times in a row. Then one ends up with two call options instead of one.

In other words, if the probability of many sequential barrier hits is large our missing correction terms will have significant value. In all other cases our formula will work fine. One solution to this is naturally to just add more and more correction terms as the probability of hitting both barriers many times sequential increases. However, intuitively this must also imply that the probability of at least one barrier hit must be very high. In that cases the value of the double barrier option must be very close to that of a plain vanilla option.

[3] Assuming flat barriers. The Ikeda and Kuintomo (1992) formula can also be used for valuation of double barrier options with curved barriers

Another observation is that our barrier approximation always will overprice double barrier in-options as long as we have even correction terms (2, 4, 6...14). The reason for this is that even correction terms implies that the last correction term always will be a long position. Again, this means that there exists a probability of ending up with two options instead of one as we should. Combining these observations we can simply increase the accuracy of our model by calculating it's value as the minimum of a plain vanilla option and the double barrier approximation with a few even correction terms, (e.g. 2, 4, or 6). Because the value of a double barrier option naturally never can be higher than a plain vanilla option this can do no harm, but only increase the accuracy. Using four correction terms the value of the double barrier in-call option can now be rewritten as

$$\begin{aligned} C_i(S,X,L,U) \approx \min\Big[& C(S,X,T); C_{ui}(S,X,U) + C_{di}(S,X,L) \\ & -\frac{X}{U}P_{ui}\left(S,\frac{U^2}{X},\frac{U^2}{L}\right) - \frac{X}{L}P_{di}\left(S,\frac{L^2}{X},\frac{L^2}{U}\right) \\ & +\frac{U}{L}C_{di}\left(S,\frac{L^2X}{U^2},\frac{L^3}{U^2}\right) + \frac{L}{U}C_{ui}\left(S,\frac{U^2X}{L^2},\frac{U^3}{L^2}\right) \\ & -\frac{LX}{U^2}P_{ui}\left(S,\frac{U^4}{L^2X},\frac{U^4}{L^3}\right) - \frac{UX}{L^2}P_{di}\left(S,\frac{L^4}{U^2X},\frac{L^4}{U^3}\right) \\ & +\frac{U^2}{L^2}C_{di}\left(S,\frac{L^4X}{U^4},\frac{L^5}{U^4}\right) + \frac{L^2}{U^2}C_{ui}\left(S,\frac{U^4X}{L^4},\frac{U^5}{L^4}\right)\Big] \end{aligned} \tag{15}$$

Using this modified version of our formula the largest mispricing in table 1 goes from -0.518185 to only -0.000024 and all the other mispricings gets even smaller. Extensive numerical investigation shows that this method is extremely accurate and robust for all types of input parameters. Even dropping the last two correction terms, numerical investigation indicates that our formula should be more than accurate enough for any practical purpos. For instance, if we now increase the volatility further in combination with longer time to maturity, and decrease the difference between the two barriers this will only increase the accuracy of our formula.

4 Conclusion

We have extended the put call transformation principle to also hold for barrier options. These new transformation relationships give new insight and are useful when calculating barrier option values. If one has a formula for a call, the relationship will give you the value for the put and vice versa. The relationships also give new opportunities for static hedging and valuation of new types of exotic options using standard single and double barrier options. Finally we have shown that the value of a double barrier option in a simple and intuitive way can be valued as the minimum of a series of a few single barrier options and a plain vanilla option.

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