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THE COLLECTOR:

Space-time Finance

The Relativity Theory's Implications for Mathematical Finance

Espen Gaarder Haug*

Abstract

Little or nothing is written about relativity theory in relation to mathematical finance. I will here explore relativity theory's implications for mathematical finance. One of the main results from my reflections on this topic is that the volatility σ is different for every observer. However, what we will call volatility-time $\sigma\sqrt{T}$ is invariant, that is the same for any observer. Further, we will see how relativity theory possibly will lead to fat-tailed distributions and stochastic volatility. Parts of the article are admittedly speculative, but not even mathematical finance can escape the fundamental laws of physics.

Keywords

Relativity, relativistic volatility, invariant uncertainty-time, from world economy to universe economy, stochastic velocity dependent volatility.

1 Introduction

The wind was blowing through my hair, I was pushing my Harley to the limit. At 120 miles per hour the 50 miles trip felt like nothing, I slowed down and stopped in front of my girlfriend. She had been waiting on the side walk with a clock we synchronized with my wristwatch just before the ride. She gave me her clock. I compared it with my wristwatch. Shit, they showed exactly the same time, not even one hundredth of a second in difference, where was the time dilation? Well this was some years ago before I understood my bike actually hardly moves and that my wristwatch was not accurate enough to measure the slight time dilation that should be there as predicted by the special theory of relativity.

Einstein's special and general relativity theories are considered among the greatest scientific discoveries of our time. Besides having changed our view of the universe, it has practical implications for nuclear physics, particle physics, navigation, metrology, geodesy, and cosmology (see Barone (1998) for more details). Strange enough, with the thousands of papers[1] and books written about relativity and its various implications little or nothing is written about its implications for mathematical finance.[2] In the Wall Street Journal November 21, 2003, I am reading about relativity and how physicists are looking at how we might travel through time. Disappointingly not even in the Wall Street Journal is there a single word on how relativity can, will and possibly already is affecting quantitative finance.

In this article I take a look at relativity theory and its implications for mathematical finance. Combining relativity theory with finance, I am naturally running the risk of being considered a crank, but what the heck—I can afford to take that chance: I'm not a Professor who has to publish in conservative academic journals (publish or perish) to keep a low-paid job.

The present theories of mathematical finance hold only for a society in which we all travel at approximately the same speed and are affected by approximately the same gravitation. It is reasonable to believe that the human race will develop fast moving space stations used for interstellar travel in the future. There is also a positive probability that we one day will find intelligent life other places in the universe where the gravitation is incredibly much higher than on earth—or maybe aliens will first will find us. Going from a world economy to a universe economy will have important implications for financial calculations, just like results for a model of a closed economy might not carry

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through to an open economy. Almost every formula and theory in mathematical finance has to be modified or generalized. Generalization of the mathematical finance theory to hold in any part of the universe at any velocity and gravitation is what I will coin Space-time Finance.

Most traders I know are typically concerned only with next minute (spot traders), a day, a week or maximum a few years in the future. When it comes to non-financial aspects of life most people appear to be interested in the time frame of a few generations at most. Developing spacecraft traveling at speeds close to the speed of light or making contact with intelligent life could easily take many more generations. For this reason I expect that this article in all likelihood will have little or no practical relevance to readers of our time. However it will hopefully be of interest to a few traders far into the future, picking up a dusty copy of Wilmott magazine. Moreover, this article could hopefully have some entertainment value for the curious mind. Without any direct comparison, recall that Bachelier (1900) theory on option pricing collected dust for more than 50 years before attracting wide attention. There are also examples in physics of crazy ideas that later made a real-world impact: In 1895 the president of the Royal Society (in science), Lord Kelvin, claimed that "heavier-than-air flying machines are impossible.'' His claim was based on our best understanding of physics at that time. Just a few years later in 1903, as we all know, the Wright brothers had achieved the "impossible." So travel at speeds significantly close to that of light may not be that far fetched after all.

With billions of galaxies, more solar systems, and probably even more planets there could easily be civilizations on other planets that are far more advanced than ours. Interestingly, some of these civilizations are possibly already using space-time finance. Not having developed colonies traveling at speeds significant to the speed of light is no excuse for us not to start developing the mathematical finance necessary for participating in a universe economy—especially considering the cost when some of us are nutty enough to consider it a fun spare time activity.

> What is the difference between reality and fiction? In fiction everything has to make sense. I will tell you about the reality.

1.1 The Special Relativity Theory

The Relativity theory is far from a one man show, even if Einstein played a major role in the development of the theory as we know it today. When Einstein wrote his 1905 paper on special relativity, the basis for his theory was already laid out by giants like Larmor, Fitzgerald, Lorentz, and Poincaré. There is no doubt that Einstein, with his very intuitive mind, came up with many key insights for the foundation of relativity theory. For example Einstein was the first to properly understand the physical implications of time dilation.[3] Lorentz himself initially did not believe in time dilation, which was a result of his own transformation[4] (Lorentz (1904)), that Einstein based much of his work on. Lorentz himself said

> "But I never thought this had anything to do with real time. . .there existed for me only one true time. I considered by time-transformation only a heuristic working hypothesis. . ."

In his 1909 paper Lorentz took time dilation seriously and formulated a relativity theory closely related to Einstein's special relativity theory. Well, more on this later.

Einstein based his special relativity theory on two postulates (Einstein 1905, Einstein 1912)

1. *Principle of special relativity:* All inertial observers are equivalent.
2. *Constancy of velocity of light:* The velocity of light is the same in all inertial systems.

Einstein accepted that the speed of light had to be constant in any frame (we will discuss this in more detail later), and he figured out that something else had to vary: time. Time dilation will play a central role in space-time finance. Even though time dilation is covered in any basic book on special relativity, we will spend some time on the basics here before we move on to space-time finance. Even before that a few basic definitions are in order:

Reference frame In most of our examples we will use two reference frames. First, a stationary inertial frame, which obeys Newton's first law of motion. Any object or body in such a frame will continue in a state of rest or with constant velocity and is not acted on by any forces external to itself. In most examples we will for simplicity assume the earth and everything on it is a stationary inertial frame. We will later loosen up on this assumption.

Second, as a moving frame we will typically use a spacecraft leaving and returning to earth. This is actually a non-inertial frame as the spacecraft must accelerate and decelerate. To begin with we will assume this is an inertial frame. We will later look at more realistic calculations where we directly take account for the acceleration.

Observer With observer we think about anyone in the same frame. This can be a person (possibly hypothetical) or a clock, or even a computer calculating the volatility of a stock.

Asset frame Where in space-time does an asset trade? One could possibly think that the properties of a financial asset are independent of where the asset trades, since it is not a physical object. This holds only because all humans at the current time are in approximately the same frame. In space-time finance the exact space-time location of the trade will have an impact. For a gold futures listed at COMEX (the metal exchange) the exact location will typically be in the trading pit in New York, Manhattan downtown. For an electronic market the trade would typically take place in a computer. The computer will be in a place in space and the trade will be executed at a given time inside the computer. Thus, any trade takes place in an exact point in space-time. After the computer accepts the trade it is too late for anyone to cancel it, even if the trader is far away and possibly not even aware if the trade has been carried out yet.

Buying or selling a securities in a location very far from you could make it difficult to communicate with each other, due to the maximum speed limit of any signal. For example how could you trade a security on earth if you lived one light year away in a space station? This could easily be solved by having someone close to the location managing your investment.

Proper time and proper volatility The proper time is the time measured by one and the same clock at the location of the events. That is we can think about a clock "attached" to the object or even the asset we are considering. For example a wristwatch worn by the same person could measure the proper time for a lifespan of this person, another name for proper time is wristwatch time. “Attaching” a clock to an asset could be done for example by measuring the time with the same computer as where the trade took place. The proper volatility of an asset will be the volatility as measured in the proper time of the asset.

2 Time Dilation

As we know from my bike ride the complexity of space-time is not apparent at low speeds. High speed velocity leads to several unexpected effects, like time dilation, length contractions, relativistic mass, and more. All these effects can be predicted using Einstein's special relativity theory. The time elapsed for a stationary observer T and a moving observer $\hat{T}$ is related by the simple, yet powerful formula

$$T = \frac{\hat{T}}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad (1)$$

where v is the velocity of the moving observer, and c the speed of light in vacuum. See appendix A for a short summary of one way to come up with this formula.

2.1 The Twin “Paradox”

Special relativity induces effects that can seem counter intuitive at first. Probably the best known of these is the twin paradox (also known as the clock paradox), see for example Taylor and Wheeler (1992), Sartori (1996) Tipler and Llewellyn (1999), Ellis and Williams (2000). As the twin paradox will play an important role in space-time finance a short introduction to the topic is in place. The twin paradox is basically about two identical twins, let's name them Tore and Kjell. Tore is leaving earth in a spacecraft that travels at a constant velocity of 80% of the speed of light, $0.8c$, to the star Alpha Centauri approximately 4.2 light years away. When the spaceship reaches Alpha Centauri it instantaneously turns and returns to earth.

The paradox arises because either twin can claim it is the other twin who is in motion relative to him. But then each twin should expect to find his twin brother younger than himself. The mistake is that we assume the situation is symmetric for the two twins. Einstein had predicted there had to be an asymmetry, and that the twin leaving in the spaceship end up being younger. In the 1950s and 1960s there was a lively discussion over the twin paradox. Philosophy Professor Dingle (1956) published a paper in Nature where he attacked Einstein's relativity theory. He claimed that the twin paradox could not be resolved and that for this reason the special relativity theory was inconsistent. Along followed a series of papers dissecting the twin paradox (see Sartori (1996) and Marder (1971) for a good reference (1971) for a good reference list). The theoretical discussion turned out in Einstein's favor.

A few years later the asymmetric solution to the twin paradox was experimentally tested. Haefele proposed flying atomic clocks around the earth, (Haefele 1970, Haefele 1971), and carried it out in collaboration with Keating, in 1971. After flying highly accurate atomic clocks around the world, they compared their readings with identical clocks left on the ground. The results where unmistakable: time ran more slowly in the airplane than in the stationary, by the exact amount predicted by Einstein's theory, (Haefele and Keating 1971b, Haefele and Keating 1971a).

Back to the twins. The twin leaving in the spacecraft has to accelerate and decelerate to get back to earth. This makes the situation asymmetric between the two twins. An observer that has to accelerate before reunion by someone that has moved at a uniform velocity (inertial frame) must have traveled faster. However the acceleration itself is not affecting time directly, only indirectly because acceleration affects velocity. This hypothesis, implicit used by Einstein in 1905, was confirmed by the famous time decay experiment on muons at CERN. The experiment accelerated the muons to 10^{18}g, and showed that all of the time dilation was due to velocity Bailey and et al. (1977). The twin paradox and its time dilation will be the foundations for much of our space-time finance. Several other experiments are consistent with the time dilation predicted by the special relativity theory.

2.2 The Current Stage of Space-time Finance

A relevant question is how fast we need to move for space-time finance to have any practical implications. The relativity theory already has practical implications on navigation, metrology, communication and cosmology. It turns out that we already today have the technology and people to conduct an experiment with measurable effects on space-time finance. The technology in question is the space shuttle. The space shuttle has a typical velocity of about 17,300 miles per hour (27,853 kph). Let us for simplicity assume a dollar billionaire got a free ticket to travel with the space shuttle. Further assume he leaves the 1 billion dollars in a bank that pays interest equivalent to 10% annually compounding, but with compounding every thousand of a second to make the calculation more accurate. The speed of the space shuttle is 7,737 meters per second. If the billionaire travels one year with the space shuttle, or 31,536,000,000 thousands of seconds, then the time gone by at earth is

$$T = \frac{31,536,000,000}{\sqrt{1 - \frac{7,737^2}{299,800,000^2}}} \approx 31,536,000,011$$

If the billionaire spends one year on earth according to his wristwatch he will receive \$100,000,000.00 in interest income, while he will receive \$100,000,000.04 in interest rate income if staying in space. That is a difference of 4 cents. This is a measurable quantity of money, but of course not economically significant,especially not for someone already a billionaire. The barrier to significant profits is that we are at a very early stage of space travel.

3 Advanced Stage of Space-time Finance

3.1 Relativistic Foreign Exchange Rates

When, and if, humans develop large spacecraft civilizations that travel at speeds significant to that of light, why not also have them develop their own currencies? We will now extend the theory of currency exchange to a world with stationary and moving civilizations.To simplify assume there are only two civilizations. One is stationary, for example earth, and one is moving relative to earth, for example a large space station. Denote the currency on the space station by Moving Currency Dollars (*MCD*) while on earth we simply assume everybody are using EURO (*EUR*). The space station has not left earth yet. Further assume the continuously compounded rate is r_m and r in the spacecraft economy and on earth respectively. The assumption of constant rates can easily be extended to stochastic interest rates. So far this is just like having two different currencies on earth. Let's say the spot currency exchange rate is quoted as *MCD* per *EUR*, $H = \frac{MCD}{EUR}$. To prevent arbitrage the forward rate F expiring at a future earth time T, must then be

$$F = He^{(r_m - r)T},$$

Assume now that the space station leaves earth at a uniform speed v to return when the currency forward expires (we are ignoring acceleration for now). It is now necessary to take into account relativistic interest rates. Denote the rate on earth as observed from the moving frame $\hat{r}$, and similarly the rate on the spacecraft as observed in the stationary frame $\hat{r}_m$. To avoid any arbitrage opportunities we must have

$$\hat{r} = \frac{r}{\sqrt{1 - \frac{v^2}{c^2}}},$$

and

$$\hat{r}_m = r_m \sqrt{1 - \frac{v^2}{c^2}}.$$

The currency forward as observed from the spacecraft time must be

$$\hat{F} = He^{(r_m - \hat{r})\hat{T}}.$$

Similarly the forward price at earth must be

$$F = He^{(\hat{r}_m - r)T},$$

which naturally implies $F = \hat{F}$ to prevent arbitrage opportunities. Similar relationships will hold between any dividend yields or cost of carry on any asset.

A special, but unlikely case is when the proper risk free rates are identical in the two economies $r_f = r$. In this case the stationary earth currency *EUR* will appreciate against the other currency. The intuition behind this is simply that if we assume the two worlds start with exactly the same resources and technology, the productivity on the moving space station will still be much lower because time and all physiological processes are slowed down. The total rate of return can of course still be higher in the space station if the rate of return is high enough to offset time dilation. The space-time equivalent rate (break-even rate) of return on the space station is simply

$$r_m = \frac{r}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Consider for instance a rate of return of 5% on earth, and that the space station moves at half the speed of light. Then the rate of return on the space station must be 5.77% per year to give the same return per year as on earth. Traveling at 98% of the speed of light the rate of return on the space station must be 25.13% to offset the time advantage (faster moving time) of the stationary civilization.

4 Space-time Uncertainty

Geometric Brownian motion assumes constant volatility. This can only be true in an inertial frame where everybody are traveling at the same speed. If we are comparing geometric Brownian motion (or any other stochastic process) in different frames then strange effects crop up.

4.1 Relativistic Uncertainty

In the case of one moving frame and one stationary frame we will no longer have one volatility for a given security, but two. If the asset trades at earth we will have

- the volatility of the asset in the stationary frame, σ, (for example earth—the proper earth volatility).
- the volatility of the earth asset as observed in the moving frame, $\hat{\sigma}$, (spacecraft).

Consider a spacecraft leaving earth at speed significant to that of light, to return at a later time. Mr. X at the spacecraft buys an option on IBM corp. that trades at one of the main exchanges on earth from Mrs. Y that lives at earth. For simplicity let us assume that the stock price in an inertial frame follows a geometric Brownian in its stationary frame on earth[5]

$$dS_t = \mu S_t dt + \sigma S_t dz.$$

In the frame of the moving observer (the spacecraft) what volatility must be observed for the stock price to make the option arbitrage free with respect to earth-inhabitants trading in the same option? The volatility measured by someone on earth is naturally σ. Let the volatility measured by someone in spacecraft time be $\hat{\sigma}$. As we already know from Einstein's theory the time measured by each observer is different. For a European option the value naturally depends on the uncertainty in form of $\sigma\sqrt{T}$ and not on σ or T independently. This holds also for American options, although it is harder to establish (a mathematical proof is given by Carr (1991)). A contingent claim will in general depend on what we will call the uncertainty-time or volatility-time, $\sigma\sqrt{T}$. To avoid any arbitrage opportunities the relationship between the volatilities as observed in two different frames must be

$$\sigma\sqrt{T} = \hat{\sigma}\sqrt{\hat{T}}$$
$$\hat{\sigma}^2 = \sigma^2 \frac{T\sqrt{1 - \frac{v^2}{c^2}}}{\hat{T}}$$
$$\hat{\sigma} = \sigma\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{4}}.$$

Similarly we can naturally have an asset trading in the moving frame. The proper volatility of that asset in it's own frame we name σ_m. The same volatility as observed from the stationary frame we name $\hat{\sigma}_m$. To avoid any arbitrage opportunities we must have

$$\sigma_m\sqrt{\hat{T}} = \hat{\sigma}_m\sqrt{T}$$
$$\hat{\sigma}_m^2 = \sigma_m^2 \frac{T\sqrt{1 - \frac{v^2}{c^2}}}{T}$$
$$\hat{\sigma}_m = \sigma_m\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{4}}.$$

These are relativistic volatilities. The geometric Brownian motion of an asset trading on earth, as observed by a moving observer, must behave according to what we will call a velocity-moved geometric Brownian motion. The various parameters in the model are shifting their value due to the velocity of the moving frame,

$$dS_{\hat{t}} = \hat{r} S_{\hat{t}} d\hat{t} + \hat{\sigma} S_{\hat{t}} dz$$

Similarly, the velocity-moved geometric Brownian motion of an asset trading in the space station, as observed by a stationary observer, must be

$$dS_t = \hat{r}_m S_t dt + \hat{\sigma}_m S_t dz$$

4.1.1 Invariant Uncertainty-time Interval

From the special relativity theory it is well known that the time interval and distances will look different for different observers, due to time dilation and length contraction. However the space-time interval[6] is invariant, i.e. the same for all observers. A similar relationship must exist when it comes to uncertainty:

> The volatility of an asset, σ and the time, T, will look different for different observers. However the uncertainty-time interval, $\sigma^2 T$, of an asset will be the same for all observers.

Invariant uncertainty-time interval is actually a condition for no arbitrage in space-time finance. The "shape" of the uncertainty-time interval can naturally be different for different stochastic processes. Instead of for instance $\sigma\sqrt{T}$ we could have a square root $\sqrt{\sigma}\sqrt{T}$, or a $\sigma^{\frac{3}{2}}\sqrt{T}$ volatility process.[7] Time and uncertainty are interrelated and can not be separated. Even if different observers observe different volatility and time for an asset trading in a given place in space and time (over time), they will all agree on the uncertainty-time. For this reason all agree on the same price for the derivative security, based on the assumption of flat space-time. In addition to velocity we must also take into account curved space-time, as we will soon do. Figure 1 illustrates relativistic time T, volatility σ, and volatility-time $\sigma\sqrt{T}$ for a security trading on earth (stationary frame), as observed from a moving frame at different velocities. The time frame is one year in stationary time. Volatility and volatility-time is measured along the left y-axis, and time against the right y-axis. Time and volatility evidently varies with the velocity, while volatility-time remains constant.

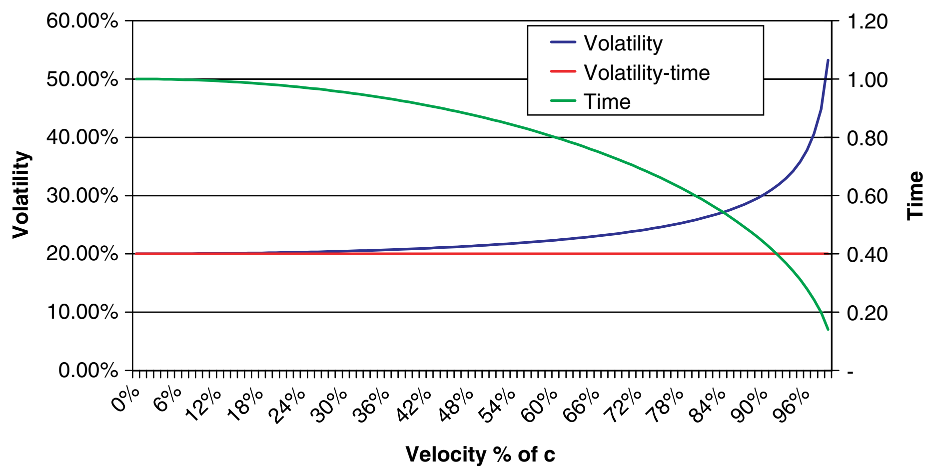


Figure 1: Relativistic Volatility

5 Is High Speed Velocity Possible?

We have seen that a trip with today's spacecraft have a measurable effect in terms of space-time finance. However, to say that the effects are economically significant would be a gross overstatement. For this to happen we need much faster means of travel.

Science fiction books and movies often involve spacecraft traveling at extremely high velocities. It is important to also have in mind that the high velocity travel must be inside the laws of physics, and it must also be physiological possible for humans to survive the trip. For example a spacecraft accelerating at 1000g would get very fast up to high speeds, but the g-force is far beyond what any human can withstand. We will here give a short summary of what actually can be possible in the future when it comes to high velocity travel. Marder (1971) discusses the theoretical and technological limits of space travel, and this section will take basis in his calculations (see also Barton (1999) and Nahin (1998)).

Let us assume we have a spaceship accelerating at 1g. As this is equivalent to the gravitation at earth such a spaceship would naturally be a very comfortable place for a human civilization. When we talk about gravitation we must be careful. We will assume the 1g gravitation is in the frame of the space traveler. From earth the gravitation of the spacecraft will be observationally different. Here the acceleration of the spacecraft will approach zero as the spacecraft approaches the speed of light. Even if the special relativity is valid only for observers moving with a constant velocity (inertial frame) this does not mean that we can not use it to predict what will happen in an accelerated frame. To do this we will make use of the "clock hypothesis". The clock hypothesis is basically a statement about the instantaneous rate of a (suitable)[8] clock depending only on its instantaneous speed. Let us define v as the speed of the spacecraft as measured from the earth frame. Further assume that at any instant there is a second system moving at a fixed speed V (earth frame) moving parallel to the spacecraft (co-moving frame). The spacecraft speed in the second system is u. If we divide the journey of the spaceship into infinitely small time steps, dt, we can assume that the change in velocity, dv, is close to zero in such a brief time interval. In other words we can still calculate the time dilation over a very short time period using the special relativity theory

$$d\hat{t} = \sqrt{1 - \frac{v^2}{c^2}}\, dt. \tag{2}$$

The interval, $\hat{T}$, of "proper time" registered by the accelerated clock in its movement between t_1 and t_2 can now simply be calculated by integrating over equation (2)

$$\hat{T} = \int d\hat{t} = \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2(t)}{c^2}}\, dt, \qquad (3)$$

where $t_2 - t_1 > \hat{T}$ is the elapsed time between two events as measured on earth (stationary frame). Furthermore, we have to be aware that the constant acceleration α, as observed on the spacecraft (the proper acceleration), will look different from earth. The acceleration in the earth frame, a is equal to (see appendix B)

$$a = c\frac{d\beta}{dt} = \alpha(1 - \beta^2)^{3/2}, \qquad (4)$$

where $\beta = \beta(t) = \frac{v}{c}$ the speed of the spacecraft in percentage of light in the earth frame. Assuming the speed of the spacecraft is zero, $\beta(0) = 0$ at the start of the journey, $t = 0$, and $\beta(t) = \beta$ we can integrate

$$\int_0^t \frac{\frac{d\beta(u)}{du}}{(1 - \beta(u)^2)^{3/2}}\, du = \int_0^t \frac{\alpha}{c}\, du,$$

this gives us

$$\frac{\beta}{\sqrt{1 - \beta^2}} = \frac{\alpha t}{c}. \qquad (5)$$

Solving for β we get

$$\beta = \frac{\alpha t / c}{\sqrt{1 + \left(\frac{\alpha t}{c}\right)^2}}. \qquad (6)$$

Assume the spaceship is leaving earth at earth time $t_1 = 0$ and spaceship time $\hat{t}_1 = 0$, that is $t_1 = \hat{t}_1$ at the start of the journey. The spaceship is constantly accelerating at 1g as measured on the space ship for $\hat{T}$ spacecraft years (T earth years). Next the spacecraft decelerates at the same rate until the spacecraft is at rest with respect to the stationary frame (earth). This means that the spacecraft travels for $2\hat{T}$ spacecraft years away from earth. The spacecraft then follows the same procedure back to earth. The whole trip takes $4\hat{T}$ spacecraft years and $4T$ earth years. The maximum velocity of the spacecraft is reached at time $\hat{T}$ and is given by replacing t by T in equation 6. We thus find that the maximum velocity as observed on earth is

$$\beta_{\max} = \frac{\alpha T / c}{\sqrt{1 + \left(\frac{\alpha T}{c}\right)^2}}. \qquad (7)$$

To find the distance as measured from earth that the spacecraft will reach we need to integrate once more

$$x(t) = \int_{t_1}^{t_2} v(t)dt = \frac{c(\sqrt{\alpha^2 t^2 + c^2} - c)}{\alpha}. \qquad (8)$$

The maximum distance in light years as measured from earth that the spacecraft will reach after $2T$ earth years ($2\hat{T}$ spacecraft years) is $2x(T)$.

We can next find the proper spacecraft time between event $t_1 = 0$ (the spacecraft leaving earth) and event t_2 (the spacecraft returning to earth) by integrating

$$\hat{T} = \int_{t_1}^{t_2} \sqrt{1 - \beta^2} dt = \int_{t_1}^{t_2} \frac{dt}{\sqrt{1 + \frac{\alpha^2 t^2}{c^2}}}, \qquad (9)$$

carrying out the integration we get

$$\hat{T} = \frac{c}{\alpha} \ln\left(\frac{\alpha T}{c} + \sqrt{1 + \frac{\alpha^2 T^2}{c^2}}\right), \qquad (10)$$

which can be simplified further to

$$\hat{T} = \frac{c}{\alpha} \sinh^{-1}\left(\frac{\alpha}{c} T\right), \qquad (11)$$

or in terms of the stationary reference frame time

$$T = \frac{c}{\alpha} \sinh\left(\frac{\alpha}{c} \hat{T}\right), \qquad (12)$$

where $\sinh()$ is the hyperbolic sine function and $\sinh^{-1}()$ is the inverse hyperbolic sine function.

Volatilities in an accelerated frame

We now have the tools to look at volatilities in an accelerated frame. The fact is that geometric Brownian motion can only exist at a constant velocity, in an inertial frame. With any form of acceleration the drift and also the volatility of the geometric Brownian motion will as a minimum be a deterministic function of the velocity, as observed from any other reference frame.

From mathematical finance it is well known that we can calculate the global volatility σ over a time period starting at t_1 ending at t_2 from a local time dependent deterministic volatility $b(t)$ by the following integral

$$\sigma^2 = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} b^2(t) dt. \qquad (13)$$

In a similar fashion we can calculate a velocity dependent deterministic volatility in an accelerating frame

$$\hat{\sigma}^2 = \frac{1}{\hat{T}} \int_{\hat{t}_1}^{\hat{t}_2} \hat{b}(\hat{t}; v)^2 d\hat{t}$$
$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{\sigma^2 dt}{\sqrt{1 + \frac{\alpha^2 t^2}{c^2}}},$$

which gives the relationship between the volatilities in the two frames

$$\hat{\sigma} = \sigma \sqrt{\frac{c}{T\alpha} \sinh^{-1}\left(\frac{T\alpha}{c}\right)},$$

or in terms of the volatility of an asset traded in a moving frame, as observed from the stationary frame,

$$\hat{\sigma}_m = \sigma_m \sqrt{\frac{c}{\hat{T}\alpha} \sinh\left(\frac{\hat{T}\alpha}{c}\right)}.$$

Numerical examples

Table 1 illustrates the consequences of a spacecraft journey accelerating at 1g. If we for instance take a look at a trip that takes ten years as measured by a wristwatch in the spacecraft, the time passed by on earth will be 24.45 years at return. The spacecraft reaches a maximum speed of 98.86% of the speed of light after 2.5 years. What a velocity! And remember, we are only at 1g! The next column tells us that over 5 years, $\hat{T} = 5$, the spacecraft will be able to travel a distance of 10.92 light years (as measured from earth). How can the spacecraft travel more than 10 light years in only 5 years? The answer is naturally because the time dilation on the spacecraft. Recall that on earth 12.71 years have gone by at the same time. The distance as measured from the spacecraft will naturally be different.

TABLE 1: SPACECRAFT ACCELERATING CONTINUOUSLY AT 1G (9.81 M/S^2), (C = 299,800,000 M/S).

Roundtrip time $4\hat{T}$	Roundtrip time $4T$	Maximum Speed % c	Distance light years	Volatility $\hat{\sigma}$	Maximum $\hat{b}(v(t))$
1	1.01	25.24%	0.065	20.111%	20.332%
2	2.09	47.46%	0.264	20.445%	21.317%
3	3.31	64.92%	0.610	21.003%	22.932%
4	4.75	77.47%	1.127	21.791%	25.151%
5	6.51	85.91%	1.849	22.815%	27.956%
10	25.43	98.86%	10.922	31.891%	51.517%
15	92.85	99.91%	44.527	49.759%	97.927%
20	337.40	99.993%	166.774	82.146%	186.599%
30	4,451.94	99.99996%	2,224.030	243.637%	677.792%
50	775,040.09	99.99999%	387,518.106	2490.044%	8943.021%

Consider next the effects on space-time finance. The relativistic volatility shows the global volatility, i.e. the “average” volatility the space crew would have measured over the whole trip for a stock that on earth had 20% constant volatility, as measured on earth.

The last column is the maximum level the instantaneously volatility attains, when the spacecraft reaches its maximum velocity.

5.1 High Velocity Spaceship

Today’s spaceships accelerate mainly using propulsion technologies. The energy content we get out of contemporary fuels is low. Uranium fission yields about 6 million times as much energy per kilogram as the burning of hydrogen. Fusion of hydrogen into helium (hydrogen bomb) yields another factor of 10. The most efficient energy for a given mass is achieved by complete annihilation of matter with antimatter. This turns all mass into energy, and gives about 140 times more energy per kilogram than hydrogen fusion. One day we might be able to have spaceships where acceleration is achieved by an matter-antimatter annihilation engine (photon engine). This is the most efficient engine we can build, and it is based on the fundamental laws of physics. For more information on this see Marder (1971).

One of the dangers with such trips would be that space is far from a perfect vacuum. Traveling through space involves frequent collisions with stray hydrogen atoms (about one for each cubic centimeter, Nahin (1998)). The result would be a high irradiation of the entire ship, with a lethal dose of gamma rays. If we ever can build protective shields for this is another questions. Luckily there is possibly an even faster and safer ways for high velocity travel, but because of limited space I will have to shun away from a discussion on this.

6 Black-Scholes in Special Relativity

In their famous price formula Black-Scholes-Merton (BSM) assumed constant volatility (geometric Brownian motion). However as we know today, the BSM formula is also fully consistent with deterministic time-varying volatility. This means that it is also consistent with a deterministic velocity-dependent volatility.

In space-time finance we will have several versions of the BSM formula. The standard form of the BSM formula is simply a special case for a inertial frame as observed from the same frame. For an option trading on the spacecraft, on a security trading at earth, in the earth currency *EUR* we will have to use the velocity-moved geometric Brownian motion as the basis for the modified BSM formula

$$dS_{\hat{t}} = \hat{r}S_{\hat{t}}d\hat{t} + \hat{\sigma}S_{\hat{t}}dz.$$

This gives us

$$c = SN(d_1) - Xe^{-\hat{r}\hat{T}}N(d_2),$$

where

$$d_1 = \frac{\ln(S/X) + (\hat{r} + \hat{\sigma}^2/2)\hat{T}}{\hat{\sigma}\sqrt{\hat{T}}},$$

and

$$d_2 = \hat{\sigma}\sqrt{\hat{T}}.$$

Similarly for an option trading at earth on an asset trading at the spacecraft we would have to replace the volatility and the risk-free rate in the Black-Scholes formula with $\hat{\sigma}_m$ and $\hat{r}_m$. Further we could have an option trading on the spacecraft in the spacecraft currency on an asset trading on earth, or an option trading on earth in the earth currency (EUR) on an asset trading on the spacecraft. This would complicate it further, but can easily be valued by making appropriate changes for relativistic effects in addition to using the well known techniques to value foreign equity options struck in domestic currency, as described in detail by Reiner (1992), see also Haug (1997).

6.1 Velocity Sensitivities

When trading options it is important to keep track of risks, efficiently summarized by the option sensitivities with respect to the key parameters, delta, gamma, vega, theta, etc. (see Haug (2003)). In the age of space-time finance it will naturally also be essential to know the derivative instruments’ sensitivities to changes in velocity. Following is the sensitivity of a stationary volatility to a small change in velocity as observed by a moving observer (here we for simplicity ignore any acceleration).

$$\frac{\partial\hat{\sigma}}{\partial v} = \frac{v\sigma}{2c^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{5}{4}}}.$$

This partial derivatives is positive and tells us that the stationary volatility as observed by a moving observer, in the moving frame, will increase the faster she travels.

The sensitivity of volatility in the moving frame to a small change in velocity, as observed by an observer in the stationary frame, is given by

$$\frac{\partial \hat{\sigma}_m}{\partial v} = -\frac{v\sigma_m}{2c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{4}}}.$$

As expected this partial derivatives is negative. This implies that uncertainty decreases the faster the moving frame moves as observed from a stationary observer.

7 Relativity and Fat Tailed Distributions

The distribution of many physical phenomena, including stock price returns and commodities, often exhibit fat tails. Empirically the returns are typically non-normally distributed, as opposed to for instance geometric Brownian motion. There can be many reasons for this phenomenon. Here we are interested in what physical laws that can induce fat tailed distributions. Assume we observe the volatility from two moving frames with different velocity, and then look at the "portfolio" volatility. If we for simplicity assume that each particle has a normal distribution, the combined distribution of the two particles will be fat tailed and leptokurtic, with Pearson kurtosis larger than 3.

7.1 Stochastic Volatility

In the real world the velocity of a single particle or a system of particles will typically change randomly over time. According to quantum mechanics and Heisenberg's uncertainty principle, it is not possible to simultaneously have perfect knowledge of both position and momentum of a particle (Heisenberg 1927). With uncertain velocity this will lead to stochastic uncertainty-time $\sigma\sqrt{T}$, where both the volatility σ and the time T will appear stochastic. More precisely , with background in relativity theory stochastic velocity v will give us apparently stochastic volatility and time (stochastic clocks). The idea of using a stochastic clock to generate stochastic volatility (uncertainty-time) is not new to finance. It dates back to the 1970 PhD thesis of now Professor Clark, later published in Econometrica, Clark (1973). Stochastic clocks have later been used as the basis for stochastic volatility models (see for instance Geman, Madan, and Yor (2000), Carr, Geman, Madan, and Yor (2003) and Carr and Wu (2004)). Instead of Brownian motion they consider stochastic time changed Brownian motion.

This literature makes few or no claims about what drives the stochastic clocks. At a recent talk at Columbia University (March 2004) Peter Carr indicated that trading volume could drive it, news coming out etc. In space-time finance we need only consider what physical laws that possibly drive stochastic clocks and volatility. We can assume velocity is stochastic and that the apparently stochastic clock simply is a deterministic function of the stochastic velocity.

Do we need to wait for the age of high velocity spacecrafts before stochastic velocity changed processes will have any practical implications on mathematical finance? There is a possibility that stochastic velocity and the relativity theory already today is what drives at least part of the stochastic volatility observed in financial markets, as well as any other stochastic uncertainty. The main question is probably if relativity here at earth has any economically significant impact on the stochastic part of volatility.

Even if we so far have assumed that the earth is an inertial frame where we all travel at approximately the same speed with respect to light, this is not true when we are moving down at particle level. All physical macroscopic objects, like people, cars, buildings are in general travel at the same speed relative to light (even including people traveling with a concord airplane). At the particle level, however, there are lots of particles traveling at very high speeds, or even at the speed of light. Particles are all the time emitting and absorbing photons, in the form of visible light or black body radiation. Every single particle, even photons in reality have their own clock. The clock in a free moving photon is from the special relativity theory "frozen": traveling at the speed of light we can cross the whole universe without any proper-time going by. A photon emitted from a particle is accelerating from zero speed to c basically instantaneously. This dose not mean we are measuring the time for every single particle here at earth, we measure the time in form of the time of macroscopic objects only, using atomic stationary clocks. This even if we all the time are affected by particles where the time and velocity are highly different from the average speed of the particles that makes up for example your body, we are not adjusting for this directly in our formulas. In the earth frame we measure the time as it was moving at one rate using atomic stationary clocks. So instead of observing stochastic clocks in other frames, they all show up as stochastic volatility, and for this reason stochastic uncertainty-time.

The number of particles that affect a stock through corporate activities etc. at any given time is just astronomical and also the number of particles affecting a stock price is varying partly randomly partly, deterministic over time. It is possible but not necessary that relativity theory for this reason to some degree can explain stochastic volatility for a security. To model the volatility of a security at particle level requires modeling the stochastic velocity and possibly other properties of each "particle" affecting the security. This requires super computers far more powerful than we can dream about today (but who knows what's around the "corner"). This would even include the physics of psychology: the human brain and its emotions that naturally lead to trades, at a quantum physics level.

8 General Relativity and Space-Time Finance

We have so far limited ourselves to the special theory of relativity. In 1916 Albert Einstein published his general relativity theory. The theory describes how gravity affects space-time. In the case of a spherical symmetric body (like the earth, sun or a black hole) Karl Schwarzschild (1873–1916) was able to come up with a beautiful closed form solution (in 1916). Just before his death, from a battle-induced illness, Karl Schwarzschild sent his closed form metric solution to Einstein. Einstein wrote to him "I had not expected that the exact solution to the problem could be formulated. Your analytic treatment of the problem appears to me splendid." Karl Schwarzschild derived his closed form metric from Einstein's field equation. Einstein's field equation is given by

$$R_{ik} - \tfrac{1}{2}Rg_{ik} - \Lambda g_{ik} = \frac{\kappa}{c^2}T_{ik},$$

where Λ is the cosmological constant, often considered to be zero, and T_{ik} is the energy momentum tensor. The κ is Einstein's gravitational constant: $\kappa = \frac{8\pi G}{2c}$ where G is the Newtonian constant of gravitation, and c is the speed of light in vacuum. In many cases Einstein's field equation is very hard to solve and often requires numerical methods. Luckily most objects in the universe are spherical and we can use the closed form Schwarzschild metric for most practical problems. In time-like form the Schwarzschild metric is given by[9]

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2 \qquad (14)$$

where τ is the proper time (wristwatch time), M is the mass of the center of attraction as measured in unites of meters, r is the reduced circumference (the circumference divided by 2π), t is the far away time, ϕ is the angle and has the same meaning in Schwarzschild geometry as it does in Euclidian geometry, and $rd\phi$ is the incremental distance measured directly along the tangent to the shell. The Schwarzschild solution gives a complete description of space-time external to a spherically symmetric, non-spinning, uncharged massive body (and everywhere around a black hole but at its central crunch point, the singularity; see Taylor and Wheeler (2000) and Misner, Thorne, and Wheeler (1973) for excellent introductions to this topic). Actually the vast majority of experimental tests of general relativity have been tests of the Schwarzschild metric. All test results have so far been consistent with Einstein's general relativity theory (?). The general relativity theory has great implications for the General Position System (GPS). The GPS system consist of multiple satellites containing atomic clocks. The elapsed time for each atomic clock has to be adjusted for both the special and general relativity. This because the effect of earth's gravity is lower far from earth and also the speed at which the satellites travel affects time. The GPS system is actively used by the military for high precision bombs, as well as for civilians like myself, to navigate the car to a new restaurant to meet up with a date.

It is the time dilation caused by gravity that is of greatest interest to space-time finance. The time-like Schwarzschild metric leads us to the formula we need:

$$d\tau = T_{shell} = T\sqrt{1 - \frac{2M}{r}}, \qquad (15)$$

where T_{shell} is the elapsed time of a clock that is at the radius r on the shell, and T is the time leaps of a far-away time. The far-away time will in practice refer to a clock that is so far away from the gravitation source that the effects of gravitation on time is insignificant compared to the calculation we are doing. From equation (15) we can see that the time at the shell (and all physical processes) will go slower than the far-away time.

Next lets take a look at a numerical example (this is also the explanations behind The Collector cartoon in this issue, as well as the animated cartoon story "Black Hole Hedge Fund"). The mass of the sun is 1477 meters, assume a black hole with 10 solar masses (14.77 kilometers). The Schwarzschild radius of the black hole is $2M = 2 \times 14770 = 29,540$ meters. This is the radius where there is no return (except for Hawking radiation), even light itself will be caught by the Schwarzschild radius. Assume a space station is hovering around the black hole at a radius of 29,552 meters. As the mass of the earth is insignificant to that of the black hole we can consider the time elapsed at earth as the far-away time. For one year passed at the space station we will have the following number of years passing by on earth

$$T_{earth} = \frac{T_{shell}}{\sqrt{1 - \frac{2M}{r}}} = \frac{1}{\sqrt{1 - \frac{29,540}{29,552}}} \approx 50$$

That is, for every year passing by in the space station 50 years will be passing by on earth. Assuming next that we place cash at earth with 10% annual return. Over 50 years on earth one million dollars will grow to

$$1,000,000 \times 1.1^{50} = 117,390,853$$

But remember only one year has gone by in the space station, so what is its equivalent annual rate of return?

$$\begin{aligned} 1,000,000 \times (1 + r) &= 117,390,853 \\ r &= \frac{117,390,853}{1,000,000} - 1 \\ &= 11,639\% \end{aligned}$$

That is the annual return on the space station is an incredible 11,639%. This explains how and why Einstein in the cartoon can promise 10,000% return from his black hole hedge fund. The additional 1,639% is simply his management fee!

Can we survive the tidal forces of high gravity

If we should ever settle down in areas with very strong gravitation it can have significant consequences for space-time finance. An important question is whether humans will be able to survive the high tidal forces one would experience there. A pilot will normally die of acceleration stress when reaching about 10g, while a quartz wristwatch will probably still continue to work normally. In 1960, astronaut Alan Shepard experienced 12g during the re-entry of the Mercury spacecraft Freedom 7 (Pickover (1996) p. 19). It is reason to believe that even moderate g-forces (2-5g) will lead to high stress on our bodies if exposed for a considerable amount of time. We have to conclude that at the current stage humans are not capable to withstand g-forces high enough to settle down in parts of the universe where the g-force is strong enough to have significant effect on space-time finance.

8.1 From Black Holes to Black-Scholes

Many years ago I got hold of the book "From Black-Scholes to Black-Holes". The book was interesting, but the title was misleading because it had nothing to do with the relationship between black holes and the formula of Black-Scholes and Merton. To use the BSM formula anywhere near a black hole we need to modify the original formula by taking into account the gravitational effects on volatility and time. Assume we are hovering around a black hole, and want to value an option trading on earth. Because the relatively much lower gravitation on earth we can assume the earth time is far away time. By some simple reflections and calculus we find that the value must be

$$c = SN(d_1) - Xe^{-\hat{r}_f T_{shell}} N(d_2), \tag{16}$$

where

$$d_1 = \frac{\ln(S/X) + (\hat{r}_f + \hat{\sigma}_f^2/2)T_{shell}}{\hat{\sigma}_f \sqrt{T_{shell}}},$$

$\hat{\sigma}_f$ is the volatility on the asset as observed from the black hole at radius r, and σ_f is the far away volatility, or basically the volatility of an asset trading at earth as observed on earth:

$$\hat{\sigma}_f = \sigma_f \left(1 - \frac{2M}{r}\right)^{-\frac{1}{4}}, \tag{17}$$

and the relationship between the far-away risk-free rate r_f and $\hat{r}_f$ as observed at the shell

$$\hat{r}_f = \frac{r_f}{\sqrt{1 - \frac{2M}{r}}}. \tag{18}$$

Similarly the volatility of an asset trading at the space station hovering around the black hole would be

$$\hat{\sigma}_{shell} = \sigma_{shell} \left(1 - \frac{2M}{r}\right)^{\frac{1}{4}}, \tag{19}$$

where σ_{shell} is the proper volatility of an asset trading at a space station hovering around the black hole at radius r, and $\hat{\sigma}_{shell}$ is the volatility of the same asset as observed far away from the gravitational field. The risk-free rate at the spacecraft as observed from a far away observer would be

$$\hat{r}_{shell} = r_{shell} \sqrt{1 - \frac{2M}{r}}. \tag{20}$$

When observing a moving frame near a gravitational field we will naturally need to take into account both the special and general relativity theory.

9 Was Einstein Right?

When it comes to the special theory of relativity was Einstein right? Many of the aspects of the relativity theory have been experimentally tested with high precision, but other aspects of his theory are not so well tested. We will here shortly mention a few of the topics where there still are untested aspects of the special and general relativity theory.

9.1 Alternatives to Einstein

Several alternative relativity theories all agree with experimental results at least as well as the special relativity theory. Out of 11 well known independent experiments said to confirm the validity of Einstein's special relativity theory non of these are able to distinguish it from for example Lorentz's relativity, Flandern (1998) or Taijii relativity, see Hsu (2000). Both Lorentz and Einstein (1905) relativity theories are based on the principle of relativity first discussed by Poincare in 1899. Lorentz assumed a universal time, a preferred frame and an luminiferous ether (a solid medium that electromagnetic waves had to travel through).

In his 1905 paper Einstein denied the existence of the luminiferous stationary ether that Lorentz believed in. Lorentz acknowledge Einstein's insight in relativity theory but never gave up on the ether theory, see Lorentz (1920). Later Einstein modified his negative attitude to the ether and developed his own theory about what he named "the new ether" and also "gravitational ether", see Einstein (1922) and Kostro (1998):

The ether is still today one of the most interesting and mysterious topics in physics. However few physicists talk about the ether, today they like to call it empty space or possibly Higgs field, see Genz (1998) for a interesting introduction to nothingness.

9.2 Constancy of the Speed of Light?

In the special relativity theory Einstein's second postulate was that the speed of light is constant in any frame. Einstein assumed clocks in an inertial frame could be synchronized with a clock at the origin of the frame by using light signals. However to synchronize the clocks in this way we must have an assumption of the speed of light. If Einstein's prescription to synchronize clocks is used, then measured speed must be the speed of light per definition. In other words the synchronization of clocks and the measured speed of light is closely connected (Reichenbach (1958). The test of the universal speed of light therefore leads to a circular argument. The currently most advanced laboratory we have for measuring the speed of light, the GPS system, confirms that the measured speed of light does not change over time or the direction of the satellite in orbit. However it cannot tell us what the speed of light is, and in particular not the one-way speed of light (Reichenbach 1958, Flandern 1998, Hsu and Zhang 2001). Long before Einstein's time, Maxwell (1831–1879) wrote

> "All methods. . .which it is practicable to determine the velocity of light from terrestrial experiments depend on the measurement of the time required for the double journey from one station to the other and back again. . ."

Reichenbach (1958), Ruderfer (1960) and others claim that Einstein's second postulate cannot even be tested. No physical experiment have been able to test if the one-way speed of light is constant in every direction (i.e. isotropic) as assumed by Einstein. There are several extensions of the special relativity theory that do not assume that the one-way speed of light is constant. They do not necessary claim that Einstein's special relativity is wrong, but that it simply is a special case of a more general relativity theory.

If the one-way speed of light is not constant, but only the two-way speed of light, the time dilation in the twin paradox will still be there, because it is based on two-way time dilation. A possibly non-isotropic one-way speed of light will naturally have an impact on time dilation during a given point on the trip, but the end result will still be the same when the stationary and the moving frame re-unites.

Already in 1898 Poincare expressed his view that the constancy of the speed of light is merely a convention. Edwards (1963) replaced Einstein's second postulate with the assumption that only the two-way (round-trip) speed of light is constant. Hsu, Leonard, and Schenble (1996) has extended the original formulations of Reichenbach's extended simultaneity and Edwards' universal two way speed of light to be consistent with 4-dimensional symmetry of the Lorentz and Poincare group. They have named this extended-relativity. After Einstein's introduction of the special relativity some physicists considered if they could construct a consistent relativity theory, based only on the first postulate. Ritz (1908), Tolman (1910), Pauli (1921) and others had to conclude that this was impossible. The reason was that they had failed to recognize the association of a 4-dimensional symmetry only associated with the first postulate. recognized this and were able to come up with a relativity theory based only on the first postulate, and that remarkably still agree with all experiments. They named this generalized relativity theory Taiji relativity. Taiji relativity does not conclude that Einstein's special relativity is wrong, it simply says that the second postulate is unnecessary to confirm experimental results. Furthermore, it shows us that our concept of time (e.g. Einstein's relativistic time or Reichenbach time) and also the speed of light in the 4-dimensional symmetry framework are human conventions rather than the inherit nature of the physical world. Special relativity, extended relativity, common relativity, and other relativity theories are typically all a special case of the generalized Taiji relativity. However Taiji relativity agrees on the experimental fact that the speed of light is independent of the source velocity (Hsu 2000, Hsu and Zhang 2001).

9.3 Faster than the Speed of Light?

In 1992 Professor Günter Nimtz and colleagues at the University of Cologne reported faster than light for microwaves using quantum tunneling. When sending light "particles" (microwaves is just another type of light/electromagnetic waves) against a thick barrier one would think all the light particles would be stopped. Quantum mechanics tells a different story. According to quantum tunneling there is always a small probability for the particle to jump through the barrier. It is like a prisoner inside his prison cell trying to walk through the wall. In our everyday life we know this is impossible, but according to quantum tunneling there is actually a positive probability that the prisoner could jump through the wall. The probability for this event is naturally incredible small. Another expert at quantum tunneling, Professor Chiao at university of Berkley at California, claimed to be able to send photons of visible light at superluminal speeds (faster than light). However he was not very concerned that this would make it possible to send information backwards in time and disrupt causality. His explanation for this is basically that the photons traveling through the barrier was random and could for this reason not be used to send any form of information.

At a physicist conference in 1995 Professor Günter Nimtz played Mozart's 40th symphony on a walkman. The bizarre part was that he claimed this was the recording from a signal sent at 4.7 times the speed of light. Many leading physicists were, and still are, very skeptical to Professors Nimtz results (Glegg 2001).[10] If these and other faster than light experiments are correct, we need to reconsider an important postulate of special relativity: that there is an absolute speed limit of light.

10 Traveling Back in Time Using Wormholes

So far we have discussed space-time finance within the limitations of traveling forward in time (time dilation). Time dilation, as predicted by the relativity theory, is as we already have mentioned confirmed by many physical experiments. If it ever will be possible to travel backwards in time is another question. One theoretical possible way to do this is by using wormholes. A wormhole is basically a short-cut through space and time. Wormholes can possibly be created by strong warping (bending) of space-time.

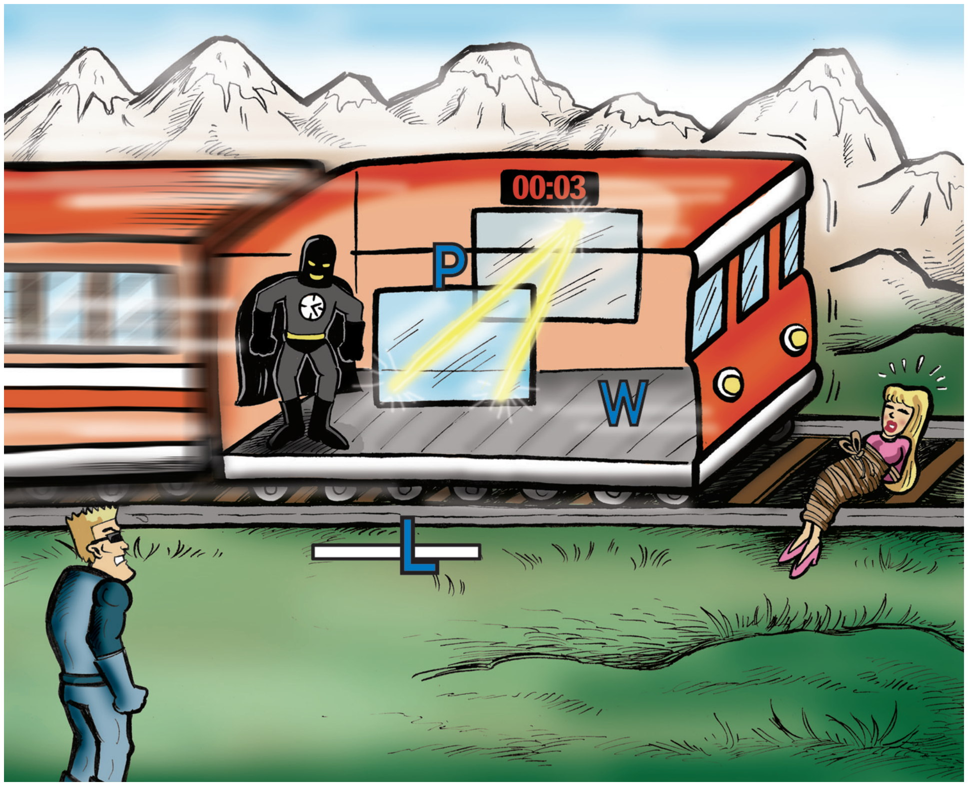
Wormhole physics can be traced back at least to the paper by Flamm (1916). In their 1935 paper Einstein and Rosen discussed a bridge across space-time, today known as an Einstein-Rosen bridge. The term "wormhole" had not yet been coined. After their 1935 paper little work were done on the subject until 20 years later when Wheeler (1955) followed up on the topic. Over the years the physicist community has come up with a variety of theoretical wormhole solutions. Most wormhole solutions are practical, however, for neither space travel nor time travel. The gravitational tidal forces will rip humans apart. They can be strong enough even to disrupt the individual nucleus of atoms. The size of some wormholes (Wheeler wormholes) are predicted to be as small as the Planck length, 10^{-35}cm. For humans to travel through such wormholes clearly seem impossible. Still, even if you cannot send a book through a telephone line you can fax its content in the form of electrons or photons. The admittedly speculative movie "Timeline" uses the same idea to "fax" humans through a tiny wormhole, basically by ripping the body apart into its sub-atomic elements (electrons, photons. . .). In this context is it worth mentioning that in 1988 Morris and Thorne came up with a theoretical wormhole system where humans could possibly pass "safe" through in a reasonable amount of time. By inducing a time-shift and bringing such wormhole mouths together, one would at least in theory create a time machine. However, in contrast to time dilation in the relativity theory there is so far no physical experiments that either can confirm or disprove the existence of wormholes.[11]

11 Conclusion

Many aspects of the relativity theory are well tested empirically. However, as we have pointed out there are still many open questions. Even if we still do not have all the answers to relativity, and therefore also not to space-time finance, I enjoy the idea that space-time finance will play some role in the future. Even quantitative finance can not escape the fundamental laws of physics.

Appendix A: Special Relativity and Time Dilation

Because many of the readers probably have little or no background in relativity theory? we will here shortly take a look at the math behind time dilation in the special relativity theory. To really exploit the special relativity theory we need the Lorentz transformation, however when it comes to the time dilation factor all we need is actually Pythagoras theorem (and some reflections that took Einstein many years). Assume that we have a moving train. Inside the moving train we have a light clock. The light clock is constructed from one mirror on each side of the inside of train.



The width between the mirrors we call w (the rest length). What is the relationship between the time it take for the light to go back and forth between the mirrors as measured on the train and on the platform?

The time measured by the moving observer, $\hat{T}$, must be

$$\hat{T} = \frac{2w}{c}.$$

The distance traveled by the light as seen from the platform, p, must be longer than the distance as measured by an observer on the train. Assuming the speed of light is constant, then more time must have passed by for the light to travel the longer path. In other words the platform time between each light clock tick is

$$T = \frac{2p}{c}. \tag{21}$$

The length p is unknown, but can easily be found by using Pythagoras theorem $a^2 = b^2 + c^2$

$$p^2 = w^2 + \left(\tfrac{1}{2}L\right)^2$$
$$p = \sqrt{w^2 + \left(\tfrac{1}{2}L\right)^2},$$

where L is the distance traveled by the train between each time the light reflect on the mirror. We know the velocity of the train is $v = L/T$ which gives $L = vT$. Replacing this measurement of p into 21 we get

$$T = \frac{2p}{c} = \frac{2\sqrt{w^2 + \left(\frac{vT}{2}\right)^2}}{c}.$$

Further we can eliminate w by replacing it with

$$w = \frac{c\hat{T}}{2}.$$

This gives

$$T^2 = \frac{4\left(\frac{1}{4}c^2\hat{T}^2 + \frac{1}{4}v^2T^2\right)}{c^2}$$
$$T^2 - \frac{v^2}{c^2}T^2 = \hat{T}^2$$
$$\hat{T} = T\sqrt{1 - \frac{v^2}{c^2}},$$

Figure 2 illustrates the classical time dilation example just given. A lot of apparently "paradoxes" can here be made, for example assume the train will reach the women in 3 seconds as measured by a clock on the train. The collector needs 4 seconds to save the women. The train is traveling at 70% of the speed of light; will the collector be able to save the women?

Appendix B: Relationship between acceleration in different frames

Here we will look at how to obtain the relationship between the acceleration of a point P moving in the x direction in an inertial system $G(x, t)$, and its acceleration in a second, parallel, system $\hat{G}(\hat{x}, \hat{t})$, which is moving with the speed V along the same direction. This section is based on Marder (1971), except I am using slightly different notation, as well as having fixed what appears to be a typo in his calculations.

Assume v is the velocity of a point in G, and u is the velocity in $\hat{G}$ then by the rules of adding velocities under special relativity we have

$$v = \frac{u + V}{1 + \frac{Vu}{c^2}},$$

and from the Lorentz transformation we have

$$T = \frac{\hat{T} + V\hat{x}/c^2}{\sqrt{1 - V^2/c^2}}.$$

Taking differentials we get

$$dv = \frac{du}{1 + \frac{Vu}{c^2}} - \frac{u + V}{\left(1 + \frac{Vu}{c^2}\right)^2}\left(\frac{V}{c^2}\right) du$$
$$= \frac{(1 - V^2/c^2)}{(1 + Vu/c^2)^2} du, \tag{22}$$

$$dt = \frac{d\hat{t}}{\sqrt{1 - V^2/c^2}}\left(1 + \frac{Vu}{c^2}\right). \tag{23}$$

Now to get the acceleration a as observed in the G frame we simply divide 22 by 23 and get

$$a = \frac{dv}{dt} = \frac{(1 - V^2/c^2)^{3/2}}{(1 + Vu/c^2)^3}\frac{du}{d\hat{t}}. \tag{24}$$

where $\frac{du}{dt}$ is the acceleration in the $\hat{G}$ frame. If $\hat{G}$ is the instantaneously co-moving system of the point P, then $u = 0$, $V = v$, and let $\alpha = \frac{du}{dt}$, then we can simplify 24 to

$$a = \frac{dv}{dt} = (1 - V^2/c^2)^{3/2}\alpha, \qquad (25)$$

which gives us the relationship between the acceleration a as observed in G and the acceleration α as observed in $\hat{G}$.

FOOTNOTES & REFERENCES

1. Already in 1922 there were more than 3400 papers written about relativity, Maurice LeCat, "Bibliographie de la Relativité," Bruxelles 1924.
2. Haug (2002) touches upon some of the relativity theory's implications for finance.
3. Larmor (1900) actually the first to introduce time dilation, but in the context of "ether theory." In his paper there is little or no discussion of the physical implications of time dilation.
4. Larmor (1900) was the first to discover the exact space-time transformation, today known as the Lorentz transformation. Lorentz probably did not know about Larmor's paper, the final rediscovery of this space-time transformation was actually done by Poincare (1905), based on Lorentz's earlier work. Lorentz himself clearly admitted this:

> "My considerations published in 1904 . . . have pushed Poincare to write this article in which he has attached my name to the transformations which I was unable to obtain. . ."

The Lorentz transformation is given by

$$\hat{x} = \frac{x - vT}{\sqrt{1 - v^2/c^2}}, \quad \hat{y} = y, \quad \hat{z} = z, \quad \hat{T} = \frac{T - vx/c^2}{\sqrt{1 - v^2/c^2}}.$$

Voigt (1887) was the first to derive a type of 4-dimensional space-time transformation, which differs slightly from the Lorentz transformation.
5. We could alternatively assume we had for instance stochastic volatility. We will look at this later in the article.
6. That the wristwatch time, a.k.a. the proper time, is invariant, independent of the reference frame, is one of the well known results from the special relativity theory.
7. The square root volatility model was to my knowledge first introduced by Heston (1993), in the form of a stochastic volatility model. Lewis (2000) is a good reference for an overview of other modeling choices for volatility.
8. For example an Einstein-Levine light clock.
9. In the space-like form the Schwarzschild metric is given by

$$d\eta^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2 d\phi^2$$

where η is the proper distance.
10. I can recommended the video "Time Travel" with interviews with both Nimtz and Chiao; by NOVA Television, BBC, 1999.
11. For a great introduction to wormholes see Thorne (1994), for a more mathematical introduction see Visser (1996).

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